

RECTILINEARIZATION OF SUB ANALYTIC SETS AS A CONSEQUENCE OF LOCAL MONOMIALIZATION

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ABSTRACT. We give a new proof of the rectilinearization theorem of Hironaka. We deduce rectilinearization as a consequence of our theorem on local monomialization for complex and real analytic morphisms.

In this paper we deduce Hironaka's rectilinearization theorem [20] as an application of our theorem on local monomialization for complex and real analytic morphisms [14].

Definition 0.1. Suppose that $\varphi : Y \rightarrow X$ is a morphism of complex or real analytic manifolds, and $p \in Y$. We will say that the map φ is monomial at p if there exist regular parameters $x_1, \dots, x_m, x_{m+1}, \dots, x_t$ in $\mathcal{O}_{X, \varphi(p)}^{\text{an}}$ and y_1, \dots, y_n in $\mathcal{O}_{Y, p}^{\text{an}}$ and $c_{ij} \in \mathbb{N}$ such that

$$\varphi^*(x_i) = \prod_{j=1}^n y_j^{c_{ij}} \text{ for } 1 \leq i \leq m$$

with $\text{rank}(c_{ij}) = m$ and $\varphi^*(x_i) = 0$ for $m < i \leq t$.

We will say that φ is monomial on Y if there exists an open cover of Y by open sets U_k which are isomorphic to open subsets of \mathbb{C}^n (or \mathbb{R}^n) and an open cover of X by open sets V_k which are isomorphic to open subsets of \mathbb{C}^t (or \mathbb{R}^t) such that $\varphi(U_k) \subset V_k$ for all k and there exist $c_{ij}(k) \in \mathbb{N}$ such that

$$\varphi^*(x_i) = \prod_{j=1}^n y_j^{c_{ij}(k)} \text{ for } 1 \leq i \leq m$$

with $\text{rank}(c_{ij}) = m$ and $\varphi^*(x_i) = 0$ for $m < i \leq t$, and where x_i and y_j are the respective coordinates on \mathbb{C}^t and \mathbb{C}^n (or \mathbb{R}^t and \mathbb{R}^n).

A local blow up of an analytic space X is a morphism $\pi : X' \rightarrow X$ determined by a triple (U, E, π) where U is an open subset of X , E is a closed analytic subspace of U and π is the composition of the inclusion of U into X with the blowup of E .

Hironaka [21] and [20] introduced the notion of an étoile on a complex analytic space Y to generalize a valuation of a function field of an algebraic variety. On any local blowup Y_1 of Y , an étoile e determines a unique point on Y_1 called the center of e . We say that a sequence of local blowups Y_1 of a complex analytic space Y is in an étoile e on Y if e has a center on Y_1 .

We now state the local monomialization theorem for complex analytic morphisms.

Theorem 0.2. (Local Monomialization, Theorem 1.2 [14]) Suppose that $\varphi : Y \rightarrow X$ is a morphism of reduced complex analytic spaces and e is an étoile over Y . Then there exists

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a commutative diagram of complex analytic morphisms

$$\begin{array}{ccc} Y_e & \xrightarrow{\varphi_e} & X_e \\ \beta \downarrow & & \downarrow \alpha \\ Y & \xrightarrow{\varphi} & X \end{array}$$

such that $\beta \in e$, the morphisms α and β are finite products of local blow ups of nonsingular analytic sub varieties, Y_e and X_e are nonsingular analytic spaces and φ_e is a monomial analytic morphism.

There exists a nowhere dense closed analytic subspace F_e of X_e such that $X_e \setminus F_e \rightarrow X$ is an open embedding and $\varphi_e^{-1}(F_e)$ is nowhere dense in Y_e .

Local monomialization theorems for real analytic morphisms are also proven in [14]. Local monomialization along an arbitrary valuation is proven for morphisms of algebraic varieties in characteristic zero in [10], [11] and [12]. Counterexamples to local monomialization for a morphism of characteristic $p > 0$ algebraic varieties is given in [13].

In Theorem 0.3 below, we show that Hironaka's rectilinearization theorem, which was originally proven in [20] can be deduced from local monomialization, Theorem 0.2. Besides local monomialization, our proof uses complexification of real analytic morphisms (Section 1 [20]), resolution of singularities of analytic spaces (Hironaka [18], [19], Aroca, Hironaka and Vicente [3] and Bierstone and Milman [6]), the Tarski Seidenberg Theorem, the fact that rectilinearization is true for semi analytic sets (this is a consequence of resolution of singularities, Hironaka [20]) and the fact that the natural map from the voûte étoilée of a complex analytic space X to X is proper (Hironaka, [21] and [20]).

Hironaka's proof (in Theorem 7.1 [20]) makes essential use of the local flattening theorem (Hironaka, Lejeune and Teissier [22] and Section 4 of [20]) and the fiber cutting lemma (Lemma 7.3.5 [20]) to reduce to consideration of proper finite morphisms. In our proof, these arguments are replaced by the local monomialization theorem, Theorem 0.2, and the reductions of Theorems 3.1 and Theorem 3.2 of this paper.

Some other notable proofs of rectilinearization are by Denef and Van Den Dries [17], Bierstone and Milman [8] and Parusinski [26]. Hironaka deduces Lojasiewicz's inequalities for sub analytic sets from rectilinearization in [20].

The related concept (to monomialization) of toroidalization for morphisms of algebraic varieties ([1] and [2]) is used by Denef to prove p -adic quantifier elimination and related results [15], [16]. Some papers on topics related to this article are Teissier [27], Cano [9], Panazzolo [25], Lichtin, [23] and [24], Belotto [4] and Belotto, Bierstone, Grandjean and Milman [5].

Theorem 0.3. (*Rectilinearization*) *Let X be a smooth connected real analytic space and let A be a sub analytic subset of X . Let $p \in X$ and let $n = \dim X$. Then there exist a finite number of real analytic morphisms $\pi_\alpha : V_\alpha \rightarrow X$ which are finite sequences of local blowups over X and induce an open embedding of an open dense subset of V_α into X such that:*

- 1) *Each V_α is isomorphic to \mathbb{R}^n ,*
- 2) *There exist compact neighborhoods K_α in V_α such that $\cup_\alpha(K_\alpha)$ is a compact neighborhood of p in X ,*
- 3) *For each α , $\pi_\alpha^{-1}(A)$ is union of quadrants in \mathbb{R}^n .*

Semi analytic and sub analytic sets in a real analytic space are defined in Section 1.

A subset B of \mathbb{R}^n is a quadrant if there exists a partition $\{1, \dots, n\} = I_0 \cup I_+ \cup I_-$ such that B is the set of $x \in \mathbb{R}^n$ such that $x_i = 0$ for all $i \in I_0$, $x_j > 0$ for all $j \in I_+$ and $x_k < 0$ for all $k \in I_-$ where x_1, \dots, x_n are the natural coordinates of x in \mathbb{R}^n .

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1. SEMI ANALYTIC AND SUB ANALYTIC SETS

We review the definitions of semi analytic and sub analytic sets from Chapter 6 [20] (also see Chapter 1 [8] or [7]).

Let X be a set and Δ be a family of subsets of X . The elementary closure $\tilde{\Delta}$ of Δ is the smallest family of subsets of X containing Δ which is stable under finite intersection, finite union and complement.

Suppose that U is an open subset of a real analytic space X .

Let $\Delta_+(U)$ be the set of subsets A of U of the form $A = \{x \in U \mid f(x) > 0\}$ for some real analytic function f on U . A subset A of X is said to be semi analytic at $x_0 \in X$ if there exists an open neighborhood U of x_0 in X such that $A \cap U$ belongs to the elementary closure of $\Delta_+(U)$. A is said to be semi analytic in X if it is semi analytic at every point of X .

Let $\Gamma(U)$ be the set of those closed subsets of U which are images of proper real analytic maps $g : Y \rightarrow U$. A subset A of X is said to be sub analytic at $x_0 \in X$ if there exists an open neighborhood U of x_0 in X such that $A \cap U$ belongs to the elementary closure of $\Gamma(U)$. A is said to be sub analytic in X if it is sub analytic at every point of X .

2. PRELIMINARIES ON ÉTOILES AND LOCAL BLOW UPS

We require that an analytic space be Hausdorff.

An étoile is defined in Definition 2.1 [21]. An étoile e over a complex analytic space X is defined as a subcategory of the category of sequences of local blow ups $\mathcal{E}(X)$ over X . If $\pi : X' \rightarrow X \in e$ then a point $e_{X'} \in X'$ is associated to e . We will call $e_{X'}$ the center of e on X' . The étoile associates a point $e_X \in X$ to X and if $\pi_1 : X_1 \rightarrow X$ is a local blow up of X such that $e_X \in X$ then $\pi_1 \in e$ and $e_{X_1} \in X_1$ satisfies $\pi_1(e_{X_1}) = e_X$. If $\pi_2 : X_2 \rightarrow X_1$ is a local blow up of X_1 such that $e_{X_1} \in X_1$ then $\pi_1 \pi_2 \in e$ and $e_{X_2} \in X_2$ satisfies $\pi_2(e_{X_2}) = e_{X_1}$. Continuing in this way, we can construct sequences of local blow ups

$$X_n \xrightarrow{\pi_n} X_{n-1} \rightarrow \dots \rightarrow X_1 \xrightarrow{\pi_1} X$$

such that $\pi_1 \dots \pi_i \in e$, with associated points $e_{X_i} \in X_i$ such that $\pi_i(e_{X_i}) = e_{X_{i-1}}$ for all i .

Let X be a complex analytic space. Let \mathcal{E}_X be the set of all étoiles over X and for $\pi : X_1 \rightarrow X$ a product of local blow ups, let

$$\mathcal{E}_\pi = \{e \in \mathcal{E}_X \mid \pi \in e\}.$$

Then the \mathcal{E}_π form a basis for a topology on \mathcal{E}_X . The space \mathcal{E}_X with this topology is called the voûte étoilée over X (Definition 3.1 [21]). The voûte étoilée is a generalization to complex analytic spaces of the Zariski Riemann manifold of a variety Z in algebraic geometry (Section 17, Chapter VI [28]).

We have a canonical map $P_X : \mathcal{E}_X \rightarrow X$ defined by $P_X(e) = e_X$ which is continuous, surjective and proper (Theorem 3.4 [21]). It is shown in Section 2 of [21] that given a

product of local blow ups $\pi : X_1 \rightarrow X$, there is a natural homeomorphism $j_\pi : \mathcal{E}_{X_1} \rightarrow \mathcal{E}_\pi$ giving a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{X_1} & \cong \mathcal{E}_\pi & \subset \mathcal{E}_X \\ P_{X_1} \downarrow & & \downarrow P_X \\ X_1 & \xrightarrow{\pi} & X. \end{array}$$

The join of $\pi_1, \pi_2 \in \mathcal{E}(Y)$ is defined in Proposition 2.9 [21]. The join is a morphism $J(\pi_1, \pi_2) : Y_J \rightarrow Y$. It has the following universal property: Suppose that $f : Z \rightarrow Y$ is a strict morphism (Definition 2.1 [21]). Then there exists a Y -morphism $Z \rightarrow Y_J$ if and only if there exist Y -morphisms $Z \rightarrow Y_1$ and $Z \rightarrow Y_2$. It follows from 2.9.2 [21] that if $\pi_1, \pi_2, e \in \mathcal{E}_Y$, then $J(\pi_1, \pi_2) \in e$. We describe the construction of Proposition 2.9 [21]. In the case when π_1 and π_2 are each local blowups, which are described by the data (U_i, E_i, π_i) , $J(\pi_1, \pi_2)$ is the blow up

$$J(\pi_1, \pi_2) : Y_J = B(\mathcal{I}_{E_1} \mathcal{I}_{E_2} \mathcal{O}_Y | U_1 \cap U_2) \rightarrow Y.$$

Now suppose that π_1 is a product $\alpha_0 \alpha_1 \cdots \alpha_r$ where $\alpha_i : Y_{i+1} \rightarrow Y_i$ are local blow ups defined by the data (U_i, E_i, α_i) , and π_2 is a product $\alpha'_0 \alpha'_1 \cdots \alpha'_r$ where $\alpha'_i : Y'_{i+1} \rightarrow Y'_i$ are local blow ups defined by the data (U'_i, E'_i, α'_i) . We may assume (by composing with identity maps) that the length of each sequence is a common value r . We define $J(\pi_1, \pi_2)$ by induction on r . Assume that $J_r = J(\alpha_0 \alpha_1 \cdots \alpha_{r-1}, \alpha'_0 \alpha'_1 \cdots \alpha'_{r-1})$ has been constructed, with projections $\gamma : Y_{J_r} \rightarrow Y_r$ and $\delta : Y_{J_r} \rightarrow Y'_r$. Then we define $J(\pi_1, \pi_2)$ to be the blow up

$$J(\pi_1, \pi_2) : Y_J = B(\mathcal{I}_{E_r} \mathcal{I}_{E'_r} \mathcal{O}_{J_r} | \gamma^{-1}(U_r) \cap \delta^{-1}(U'_r)) \rightarrow Y.$$

Suppose that $\varphi : X \rightarrow Y$ is a morphism of complex analytic spaces, and $\pi : Y' \rightarrow Y \in \mathcal{E}(Y)$. The morphism $\varphi^{-1}[\pi] : \varphi^{-1}[Y'] \rightarrow X$ will denote the strict transform of φ by π (Section 2 of [22]).

In the case of a single local blowup (U, E, π) of Y , $\varphi^{-1}[Y']$ is the blow up $B(\mathcal{I}_E \mathcal{O}_X | \varphi^{-1}(U))$. In the case when $\pi = \pi_0 \pi_1 \cdots \pi_r$ with $\pi_i : Y_{i+1} \rightarrow Y_i$ given by local blow ups (U_i, E_i, π_i) , we inductively define $\varphi^{-1}[\pi]$. Assume that $\pi^{-1}[\pi_0 \cdots \pi_{r-1}]$ has been constructed. Let $h = \pi_0 \cdots \pi_{r-1}$, so that $\pi = h \pi_r$. Let $\varphi' : \varphi^{-1}[Y_r] \rightarrow Y_r$ be the natural morphism. Then define $\varphi^{-1}[Y_{r+1}]$ to be the blow up $B(\mathcal{I}_{E_r} \mathcal{O}_{\varphi^{-1}[Y_r]} | (\varphi')^{-1}(U_r))$.

3. RECTILINEARIZATION

In this section we prove rectilinearization, Theorem 3.5. We use the method of complexification of a real analytic morphism (Section 1, [20]).

Theorem 3.1. *Suppose that $\varphi : Y \rightarrow X$ is a morphism of reduced complex analytic spaces, K is a compact neighborhood in Y and f is an étoile over X . Then there exist local monomializations*

$$(1) \quad \begin{array}{ccc} Y_i & \xrightarrow{\varphi_i} & X_i \\ \delta_i \downarrow & & \downarrow \gamma_i \\ Y & \xrightarrow{\varphi} & X \end{array}$$

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for $1 \leq i \leq r$ and $\pi : X_f \rightarrow X \in f$ such that there are commutative diagrams for $1 \leq i \leq t \leq r$

$$(2) \quad \begin{array}{ccc} \overline{Y}_i & \xrightarrow{\tau_i} & X_f \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y_i & \xrightarrow{\varphi_i} & X_i \\ \delta_i \downarrow & & \downarrow \gamma_i \\ Y & \xrightarrow{\varphi} & X \end{array}$$

where $\overline{Y}_i = \varphi_i^{-1}[X_f]$. Let $\psi_i = \delta_i \beta_i$ and $\pi = \gamma_i \alpha_i$. There exists a closed analytic subspace G_f of X_f which is nowhere dense in X_f such that $X_f \setminus G_f \rightarrow X$ is an open embedding, $\pi^{-1}(\pi(G_f)) = G_f$, the vertical arrows are products of a finite number of local blow ups of smooth subspaces and

$$\cup_{i=1}^t \alpha_i^{-1}(\varphi_i(\delta_i^{-1}(K))) \setminus G_f = \cup_{i=1}^t \tau_i(\psi_i^{-1}(K)) \setminus G_f = \pi^{-1}(\varphi(K)) \setminus G_f.$$

There exists a compact neighborhood L of f_{X_f} in X_f , a morphism of reduced complex analytic spaces $u : Z \rightarrow X$ and a compact neighborhood M in Z such that $\dim Z < \dim Y$, $u(Z) \subset \varphi(Y)$ and $\varphi(K) \cap \pi(G_f \cap L) = u(M)$.

Proof. By Theorem 0.2, we may construct local monomializations (1), with relatively compact open neighborhoods C_i in Y_i with closures K_i such that $\{\mathcal{E}_{C_1}, \dots, \mathcal{E}_{C_r}\}$ is an open cover of the compact set $\rho_Y^{-1}(K)$ (ρ_Y is proper by Theorem 3.16 [20]) and γ_i are sequences of local blow ups of smooth subspaces. We have that

$$(3) \quad K \subset \cup_{i=1}^r \delta_i(C_i).$$

Further, there exist closed analytic subspaces G_i of X_i which are nowhere dense in X_i such that $X_i \setminus G_i \rightarrow X$ is an open embedding and $\varphi_i^{-1}(G_i)$ is nowhere dense in Y_i . Reindex the diagrams (1) so that $f \in \mathcal{E}_{X_i}$ if $1 \leq i \leq t$ and $f \notin \mathcal{E}_{X_i}$ if $t < i \leq r$. Suppose that i is an index such that $f \notin \mathcal{E}_{X_i}$ (that is, $t < i \leq r$). The morphism $X_i \rightarrow X$ has a factorization

$$X_i = V_n \xrightarrow{\sigma_n} \dots \rightarrow V_2 \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V_0 = X$$

where each $\sigma_j : V_j \rightarrow V_{j-1}$ is a local blowup (U_j, E_j, σ_j) . There exists a smallest j such that $f_{V_{j-1}} \notin U_j$.

Let X_i^* be an open neighborhood of $f_{V_{j-1}}$ in V_{j-1} which is disjoint from U_j . Then $f \in \mathcal{E}_{X_i^*}$ and $\mathcal{E}_{X_i^*} \cap \mathcal{E}_{X_i} = \emptyset$. Let $\pi : X_f \rightarrow X$ be a (global) resolution of singularities of the join of the X_i which satisfy $f \in \mathcal{E}_{X_i}$ and of the X_i^* such that $f \notin \mathcal{E}_{X_i}$ and so that $\pi : X_f \rightarrow X \in f$ is a sequence of local blowups whose centers are nonsingular and such that $\alpha_i^{-1}(\varphi_i(Y_i))$ is nowhere dense in X_f for all i . Then, $\mathcal{E}_{X_f} \subset \mathcal{E}_{X_i}$ if $f \in \mathcal{E}_{X_i}$ and $\mathcal{E}_{X_f} \cap \mathcal{E}_{X_i} = \emptyset$ if $f \notin \mathcal{E}_{X_i}$. We have factorizations

$$X_f \xrightarrow{\alpha_i} X_i \xrightarrow{\gamma_i} X$$

of π if $f \in \mathcal{E}_{X_i}$. For $1 \leq i \leq t$ let

$$\begin{array}{ccc} \overline{Y}_i & \xrightarrow{\tau_i} & X_f \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y_i & \xrightarrow{\varphi_i} & X_i \end{array}$$

be the natural commutative diagram of morphisms, with $\overline{Y}_i = \varphi_i^{-1}[X_f]$. Let $\psi_i = \delta_i \beta_i$. Let G_f be the union of the preimages of the subspaces blown up in a factorization of π by local blowups. Then G_f is a nowhere dense closed analytic subset of X_f such that $X_f \setminus G_f \rightarrow X$ is an open embedding and $\pi^{-1}(\pi(G_f)) = G_f$. Further, $\tau_i^{-1}(G_f)$ is nowhere

dense in \overline{Y}_i for all i . Let $U = X_f \setminus G_f$. Suppose that $q \in \varphi(K) \cap \pi(U)$. There there exists $p \in K$ such that $\varphi(p) = q$ and there exists i and $p' \in C_i$ such that $\delta_i(p') = p$ by (3). Let $q' = \varphi_i(p') \in X_i$. There exists $\lambda \in \mathcal{E}_X$ such that $\lambda_{X_i} = q'$ and thus $\lambda_X = q$. Since $q \in \pi(U)$, we can regard q as an element of X_f with $\lambda_{X_f} = q$. Thus $\lambda \in \mathcal{E}_{X_f}$ so that $\lambda \in \mathcal{E}_{X_f} \cap \mathcal{E}_{X_i}$, and so $f \in \mathcal{E}_{X_i}$ as this intersection is nonempty. We have that $\varphi_i(p') = q'$ and $\alpha_i : X_f \rightarrow X_i$ is an open embedding in a neighborhood of q , so β_i is an open embedding in a neighborhood of p' . Thus $q = \lambda_{X_f} \in \tau_i(\psi_i^{-1}(K))$. Whence

$$\pi^{-1}(\varphi(K)) \cap U \subset \cup_i \tau_i(\beta_i^{-1}(C_i \cap \delta_i^{-1}(K))) \cap U \subset \cup_i \tau_i(\psi_i^{-1}(K)) \cap U.$$

We have

$$\cup_i \tau_i(\psi_i^{-1}(K)) \subset \pi^{-1}(\varphi(K))$$

since

$$\pi \tau_i(\psi_i^{-1}(K)) = \varphi \psi_i(\psi_i^{-1}(K)) \subset \varphi(K)$$

for $1 \leq i \leq t$. Thus

$$(4) \quad \cup_i \tau_i(\psi_i^{-1}(K)) \cap U = \pi^{-1}(\varphi(K)) \cap U.$$

Let V_f be a relatively compact open neighborhood of f_{X_f} in X_f . Let L be the closure of V_f in X_f . Then $\beta_i^{-1}(C_i) \cap \tau_i^{-1}(V_f)$ are relatively compact open subsets of \overline{Y}_i with closures $L_i = \beta_i^{-1}(K_i) \cap \tau_i^{-1}(L)$ for $1 \leq i \leq t$. Further,

$$\cup_i \varphi \psi_i(\psi_i^{-1}(K) \cap L_i) \subset \pi(L) \cap \varphi(K)$$

and so

$$\cup_i \varphi \psi_i(\psi_i^{-1}(K) \cap L_i) \setminus \pi(G_f) = (\pi(L) \setminus \pi(G_f)) \cap \varphi(K)$$

by (4). For all i , the compact set $\pi(G_f) \cap [\varphi \psi_i(\psi_i^{-1}(K) \cap L_i)]$ is nowhere dense in the compact set $\varphi \psi_i(\psi_i^{-1}(K) \cap L_i)$ since $\tau_i^{-1}(G_f) \cap \psi_i^{-1}(K) \cap L_i$ is nowhere dense in the compact neighborhood $\psi_i^{-1}(K) \cap L_i$ in Y_i .

Thus the compact set $\cup_i \varphi \psi_i(\psi_i^{-1}(K) \cap L_i)$ is everywhere dense in the compact set $\pi(L) \cap \varphi(K)$. Thus

$$\cup_i \varphi \psi_i(\psi_i^{-1}(K) \cap L_i) = \pi(L) \cap \varphi(K)$$

and so

$$\cup_i \varphi \varphi_i(\psi_i^{-1}(K) \cap L_i \cap \tau_i^{-1}(G_f)) = \pi(G_f \cap L) \cap \varphi(K).$$

Let $Z = \coprod_{1 \leq i \leq t} \tau_i^{-1}(G_f)$ be the disjoint union of the analytic spaces $\tau_i^{-1}(G_f)$ with associated morphism $u = \coprod_i \varphi \psi_i : Z \rightarrow X$ and compact subset $M = \coprod_i \psi_i^{-1}(K) \cap L_i \cap \tau_i^{-1}(G_f)$ of Z .

Then $\dim Z < \dim Y$, $u(Z) \subset \varphi(Y)$ and $\varphi(K) \cap \pi(G_f \cap L) = u(M)$. \square

Suppose $\varphi : Y \rightarrow X$ is a morphism of reduced real analytic spaces such that X is smooth. Let $\tilde{Y} \rightarrow \tilde{X}$ be a complexification of $\varphi : Y \rightarrow X$ such that \tilde{X} is smooth and \tilde{Y} is reduced. Suppose that \tilde{K} is a compact neighborhood in \tilde{Y} which is invariant under the auto conjugation of \tilde{Y} . Let K be the real part of \tilde{K} , which is a compact neighborhood in Y . Let

$$\tilde{Y} = \tilde{Y}^{(n)} \supset \tilde{Y}^{(n-1)} \supset \dots \supset \tilde{Y}^{(0)} = \emptyset$$

be the stratification of \tilde{Y} where $\tilde{Y}^{(i-1)} = \text{sing}(\tilde{Y}^{(i)})$ is the singular locus of $\tilde{Y}^{(i)}$, and let

$$Y = Y^{(n)} \supset Y^{(n-1)} \supset \dots \supset Y^{(0)}$$

be the induced smooth real analytic stratification of Y . We have induced compact neighborhoods $\tilde{K} \cap Y^{(i)}$ in $Y^{(i)}$, with $K = \tilde{K} \cap Y^{(n)}$. There exist global resolutions of singularities $\tilde{\lambda}_i : (\tilde{Y}^{(i)})^* \rightarrow \tilde{Y}^{(i)}$ which have an auto conjugation such that the real part of $\tilde{\lambda}_i : (\tilde{Y}^{(i)})^* \rightarrow \tilde{Y}^{(i)}$ is $\lambda_i : (Y^{(i)})^* \rightarrow Y^{(i)}$ where $(Y^{(i)})^*$ is smooth (Desingularization I, 5.10 [20]). The morphism $\tilde{\lambda}_i$ is proper, so $\tilde{K}_i = \tilde{\lambda}_i^{-1}(K \cap \tilde{Y}^{(i)})$ is a compact neighborhood in $\tilde{Y}^{(i)}$ with compact real neighborhood $K_i = \lambda_i^{-1}(K \cap Y^{(i)})$ in $Y^{(i)}$.

Let $Y' = \coprod_i (Y^{(i)})^*$. We have that the induced morphism $\varphi^* : Y' \rightarrow Y$ is proper and surjective. Let $K' = (\lambda^*)^{-1}(K)$, a compact neighborhood in Y' . Let $\tilde{Y}' = \coprod_i (\tilde{Y}^{(i)})^*$ with induced complex analytic morphism $\tilde{\lambda}^* : \tilde{Y}' \rightarrow \tilde{Y}$. Then $\tilde{\lambda}^* : \tilde{Y}' \rightarrow \tilde{Y}$ is a complexification of $\lambda^* : Y' \rightarrow Y$. Let $\tilde{K}' = (\tilde{\lambda}^*)^{-1}(K)$, which is a compact neighborhood in \tilde{Y}' with $(\tilde{K}') \cap Y' = K'$.

By Theorem 3.1, applied to the complex analytic morphism $\tilde{\varphi}\tilde{\lambda}^* : \tilde{Y}' \rightarrow \tilde{X}$, the compact neighborhood \tilde{K}' in \tilde{Y}' and an étoile f over \tilde{X} , there exist commutative diagrams

$$\begin{array}{ccc} \tilde{Y}_i & \xrightarrow{\tilde{\tau}_i} & \tilde{X}_f \\ \tilde{\beta}_i \downarrow & & \downarrow \tilde{\alpha}_i \\ \tilde{Y}_i & \xrightarrow{\tilde{\varphi}_i} & \tilde{X}_i \\ \tilde{\delta}_i \downarrow & & \downarrow \tilde{\gamma}_i \\ \tilde{Y}' & \xrightarrow{\tilde{\varphi}\tilde{\lambda}^*} & \tilde{X} \\ \searrow & & \nearrow \\ & \tilde{Y} & \end{array}$$

and a closed analytic subspace \tilde{G}_f of \tilde{X}_f such that

$$(5) \quad \cup_{i=1}^t \tilde{\tau}_i(\tilde{\psi}_i^{-1}(\tilde{K}')) \setminus \tilde{G}_f = \tilde{\pi}^{-1}(\tilde{\varphi}\tilde{\lambda}^*(\tilde{K}')) \setminus \tilde{G}_f$$

and there exists a compact neighborhood \tilde{L} of $f_{\tilde{X}_f}$ in \tilde{X}_f , a morphism of reduced complex analytic spaces $\tilde{u} : \tilde{Z} \rightarrow \tilde{X}$ and a compact neighborhood \tilde{M} in \tilde{Z} such that $\dim \tilde{Z} < \dim \tilde{Y}$ and

$$\tilde{\varphi}\tilde{\lambda}^*(\tilde{K}') \cap \tilde{\pi}(\tilde{G}_f \cap \tilde{L}) = \tilde{u}(\tilde{M}).$$

We can construct the above complex analytic spaces and morphisms so that there are compatible auto conjugations which preserve \tilde{G}_f , \tilde{Z} , \tilde{L} and \tilde{M} and so that the real part X_f of \tilde{X}_f is nonempty if and only if $f_{\tilde{X}_f}$ is a real point (by Theorems 8.4 and 8.5 [14]).

Taking the invariants of the auto conjugations, we thus have whenever $X_f \neq \emptyset$, induced commutative diagrams of real analytic spaces and morphisms

$$\begin{array}{ccc} \overline{Y}_i & \xrightarrow{\tau_i} & X_f \\ \beta_i \downarrow & & \downarrow \alpha_i \\ Y_i & \xrightarrow{\varphi_i} & X_i \\ \delta_i \downarrow & & \downarrow \gamma_i \\ Y' & \xrightarrow{\varphi\lambda^*} & X \\ \searrow & & \nearrow \\ & Y & \end{array}$$

with a closed real analytic subspace $G_f = \tilde{G}_f \cap X_f$ of X_f . We have that G_f is nowhere dense in X_f since X_f is smooth (for instance by Lemma 8.2 [14]), and thus $\dim G_f < \dim X_f = \dim Y$.

Also, taking the real part of $\tilde{u} : \tilde{Z} \rightarrow \tilde{Y}'$, we have a morphism of reduced real analytic spaces $u : Z \rightarrow Y'$ and a compact subset M of Z such that $\dim Z < \dim Y$. The analog of (3) in Theorem 3.1, $K' \subset \cup_{i=1}^r \delta_i(C_i)$ where C_i is the real part of \tilde{C}_i , is true as Y' is smooth (by Theorem 8.7 [14] and its proof). Then the argument following (3) in Theorem 3.1 shows that

$$\cup_{i=1}^t \tau_i(\psi_i^{-1}(K')) \setminus G_f = \pi^{-1}(\varphi \lambda^*(K')) \setminus G_f = \pi^{-1}(\varphi(K)) \setminus G_f$$

and

$$\pi(K) \cap \pi(G_f \cap L) = \varphi \lambda^*(K') \cap \pi(G_f \cap L) = u(M).$$

Theorem 3.2. *Suppose that $\varphi : Y \rightarrow X$ is a morphism of reduced complex analytic spaces, $K \subset Y$ is a compact neighborhood in Y and $h \in \mathcal{E}_X$. Then there exists $d_h : X_h \rightarrow X \in h$, morphisms of reduced complex analytic spaces $\varphi_i : Y_i \rightarrow X$ for $0 \leq i \leq t$ with compact neighborhoods K_i in Y_i such that $\varphi_0 = \varphi$, $Y_0 = Y$, $K_0 = K$, $\varphi_{i+1}(Y_{i+1}) \subset \varphi_i(Y_i)$, $\dim Y_{i+1} < \dim Y_i$ for all i and $Y_t = \emptyset$. There exist commutative diagrams for $0 \leq i \leq t$*

$$(6) \quad \begin{array}{ccc} \hat{Y}_{ij} & \xrightarrow{\sigma_{ij}} & X_h \\ b_{ij} \downarrow & & \downarrow a_i \\ \bar{Y}_{ij} & \xrightarrow{\tau_{ij}} & X_i \\ \beta_{ij} \downarrow & & \downarrow \alpha_{ij} \\ Y_{ij} & \xrightarrow{\varphi_{ij}} & X_{ij} \\ \delta_{ij} \downarrow & & \downarrow \gamma_{ij} \\ Y_i & \xrightarrow{\varphi_i} & X \end{array}$$

where $\varphi_{ij} : Y_{ij} \rightarrow X_{ij}$ are monomial morphisms, $\bar{Y}_{ij} = \varphi_{ij}^{-1}[X_i]$ and $\hat{Y}_{ij} = \tau_{ij}^{-1}[X_h]$, $\psi_{ij} = \delta_{ij}\beta_{ij}$, $c_{ij} = \psi_{ij}b_{ij}$, $\pi_i = \gamma_{ij}\alpha_{ij}$, $\varepsilon_{ij} = \alpha_{ij}a_i$ and $d_h = \pi_i a_i$ such that

$$(7) \quad d_h^{-1}(\varphi(K)) = \cup_i a_i^{-1}[\cup_j \tau_{ij}(\psi_{ij}^{-1}(K_i))] = \cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(\delta_{ij}^{-1}(K)))$$

Proof. We construct commutative diagrams

$$\begin{array}{ccc} \bar{Y}_{ij} & \rightarrow & X_i \\ \downarrow & & \downarrow \\ Y_{ij} & \rightarrow & X_{ij} \\ \downarrow & & \downarrow \\ Y_i & \rightarrow & X \end{array}$$

satisfying the conclusions of the theorem by induction on i , using Theorem 3.1. In particular, there exist nowhere dense closed analytic subsets G_i of X_i such that $\pi_i^{-1}(\pi_i(G_i)) = G_i$ for all i and $X_i \setminus G_i \rightarrow X$ is an open embedding and there exist compact neighborhoods K_i in Y_i and L_i of f_{X_i} in X_i such that $X_t = \emptyset$, and for all i ,

$$(8) \quad \cup_j \tau_{ij}(\psi_{ij}^{-1}(K_i)) \setminus G_i = \pi_i^{-1}(\varphi_i(K_i)) \setminus G_i$$

and

$$(9) \quad \varphi_i(K_i) \cap \pi_i(G_i \cap L_i) = \varphi_{i+1}(K_{i+1}).$$

We then have (by the definition of an étoile) that there exists $X_h \rightarrow X \in h$ such that we have a commutative diagram (6) such that $a_i(X_h) \subset L_i$ for all i . For all i , (8) implies

$$(10) \quad a_i^{-1}[\cup_j \tau_{ij}(\psi_{ij}^{-1}(K_i))] \cap (X_f \setminus a_i^{-1}(G_i)) = d_h^{-1}(\varphi_i(K_i)) \cap (X_f \setminus a_i^{-1}(G_i)).$$

Now, (9) implies

$$\pi_{i-1}^{-1}(\varphi_{i-1}(K_{i-1})) \cap G_{i-1} \cap \pi_{i-1}^{-1}(\pi_{i-1}(L_{i-1})) = \pi_{i-1}^{-1}(\varphi_i(K_i))$$

for all i , and so since $a_{i-1}(X_h) \subset L_{i-1}$,

$$d_h^{-1}(\varphi_{i-1}(K_{i-1})) \cap a_{i-1}^{-1}(G_{i-1}) = d_h^{-1}(\varphi_i(K_i))$$

for all i . Thus

$$(11) \quad d_h^{-1}(\varphi(K)) \cap a_0^{-1}(G_0) \cap \cdots \cap a_{i-1}^{-1}(G_{i-1}) = d_h^{-1}(\varphi_i(K_i)).$$

Since $G_t = \emptyset$, and we certainly have

$$\cup_i a_i^{-1}[\cup_j \tau_{ij}(\psi_{ij}^{-1}(K_i))] \subset d_h^{-1}(\varphi(K)),$$

(7) follows from induction on i , using (10) and (11) and since

$$\cup_{i=0}^t [a_0^{-1}(G_0) \cap \cdots \cap a_{i-1}^{-1}(G_{i-1})] \cap (X_f \setminus a_i^{-1}(G_i)) = X_f.$$

□

From the discussion after Theorem 3.1 and Theorem 3.2, we obtain the following statement. Suppose $\varphi : Y \rightarrow X$ is a morphism of reduced real analytic spaces such that X is smooth. Let $\tilde{Y} \rightarrow \tilde{X}$ be a complexification of $\varphi : Y \rightarrow X$ such that \tilde{X} is smooth and \tilde{Y} is reduced. Suppose that \tilde{K} is a compact neighborhood in \tilde{Y} which is invariant under the auto conjugation of \tilde{Y} . Let K be the real part of \tilde{K} which is a compact neighborhood in Y .

Suppose that $h \in \mathcal{E}_X$. Then there exists $\tilde{d}_h : \tilde{X}_h \rightarrow \tilde{X} \in h$, morphisms of reduced complex analytic spaces $\tilde{\varphi}_i : \tilde{Y}_i \rightarrow \tilde{X}$ for $0 \leq i \leq t$ with compact neighborhoods \tilde{K}_i in \tilde{Y}_i such that $\tilde{\varphi}_0(\tilde{K}_0) = \tilde{\varphi}(\tilde{K})$, $\dim \tilde{Y}_{i+1} < \dim \tilde{Y}_i$ for all i and $\tilde{Y}_t = \emptyset$. There exist commutative diagrams for $0 \leq i \leq t$

$$(12) \quad \begin{array}{ccc} \tilde{Y}_{ij} & \xrightarrow{\tilde{\sigma}_{ij}} & \tilde{X}_h \\ \tilde{b}_{ij} \downarrow & & \downarrow \tilde{a}_i \\ \tilde{Y}_{ij} & \xrightarrow{\tilde{\tau}_{ij}} & \tilde{X}_i \\ \beta_{ij} \downarrow & & \downarrow \alpha_{ij} \\ \tilde{Y}_{ij} & \xrightarrow{\tilde{\varphi}_{ij}} & \tilde{X}_{ij} \\ \tilde{\delta}_{ij} \downarrow & & \downarrow \tilde{\gamma}_{ij} \\ \tilde{Y}_i & \xrightarrow{\tilde{\varphi}_i} & \tilde{X} \end{array}$$

where $\varphi_{ij} : \tilde{Y}_{ij} \rightarrow \tilde{X}_{ij}$ are monomial morphisms, $\tilde{Y}_{ij} = \tilde{\varphi}_{ij}^{-1}[X_i]$ and $\tilde{Y}_{ij} = \tilde{\tau}_{ij}^{-1}[\tilde{X}_h]$. Let $\tilde{\psi}_{ij} = \tilde{\delta}_{ij}\tilde{\beta}_{ij}$, $\tilde{c}_{ij} = \tilde{\psi}_{ij}\tilde{b}_{ij}$, $\tilde{\pi}_i = \tilde{\gamma}_{ij}\tilde{\alpha}_{ij}$ and $\tilde{d}_h = \tilde{\pi}_i\tilde{a}_i$ such that

$$(13) \quad \tilde{d}_h^{-1}(\tilde{\varphi}(\tilde{K})) = \cup_{i,j} \tilde{\varepsilon}_{ij}^{-1}(\tilde{\varphi}_{ij}(\tilde{\delta}_{ij}^{-1}(\tilde{K}))) = \cup_i \tilde{a}_i^{-1}[\cup_j \tilde{\tau}_{ij}(\tilde{\psi}_{ij}^{-1}(\tilde{K}_i))]$$

Further, there are compatible auto conjugations of these analytic spaces and morphisms such that the real parts are $d_h : X_h \rightarrow X$, morphisms of reduced real analytic spaces $\varphi_i : Y_i \rightarrow X$ for $0 \leq i \leq t$ with compact neighborhoods K_i in Y_i with $K_i = \tilde{K}_i \cap Y_i$ such that $\varphi_0(K_0) = \varphi(K)$, $\dim Y_{i+1} < \dim Y_i$ for all i and $Y_t = \emptyset$. We may assume that $X_h \neq \emptyset$ if and only if $h_{\tilde{X}_h}$ is a real point of \tilde{X}_h . Suppose that $X_h \neq \emptyset$. Then there exist

commutative diagrams for $0 \leq i \leq t$

$$(14) \quad \begin{array}{ccc} \hat{Y}_{ij} & \xrightarrow{\sigma_{ij}} & X_h \\ b_{ij} \downarrow & & \downarrow a_i \\ \bar{Y}_{ij} & \xrightarrow{\tau_{ij}} & X_i \\ \beta_{ij} \downarrow & & \downarrow a_{ij} \\ Y_{ij} & \xrightarrow{\varphi_{ij}} & X_{ij} \\ \delta_{ij} \downarrow & & \downarrow \gamma_{ij} \\ Y_i & \xrightarrow{\varphi_i} & X \end{array}$$

where $\varphi_{ij} : Y_{ij} \rightarrow X_{ij}$ are monomial morphisms, $\bar{Y}_{ij} = \varphi_{ij}^{-1}[X_i]$ and $\hat{Y}_{ij} = \tau_{ij}^{-1}[X_h]$, $\psi_{ij} = \delta_{ij}\beta_{ij}$, $c_{ij} = \psi_{ij}b_{ij}$, $\pi_i = \gamma_{ij}\alpha_{ij}$ and $d_h = \pi_i a_i$ such that

$$(15) \quad d_h^{-1}(\varphi(K)) = \cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(\delta_{ij}^{-1}(K))) = \cup_i a_i^{-1}[\cup_j \tau_{ij}(\psi_{ij}^{-1}(K_i))]$$

Theorem 3.3. *Suppose that X and Y are real analytic spaces such that X is smooth and $\varphi : Y \rightarrow X$ is a proper real analytic map. Let $p \in X$. Then there exists a finite number of real analytic maps $\pi_\alpha : V_\alpha \rightarrow X$ such that:*

- 1) *Each V_α is smooth and each π_α is a composition of local blowups of nonsingular sub varieties,*
- 2) *There exist compact neighborhoods N_α in V_α for all α such that $\cup_\alpha \pi_\alpha(N_\alpha)$ is a compact neighborhood of p in X ,*
- 3) *For all α , $\pi_\alpha^{-1}(\varphi(Y))$ is a semi analytic subset of V_α .*

Proof. Let $\tilde{\varphi} : \tilde{Y} \rightarrow \tilde{X}$ be a complexification of $\varphi : Y \rightarrow X$ so that \tilde{X} is smooth and \tilde{Y} is reduced.

Let \tilde{U} be a relatively compact open neighborhood of p in \tilde{X} which is invariant under the auto conjugation of \tilde{X} and let \tilde{L} be the closure of \tilde{U} in \tilde{X} . Let $L = \tilde{L} \cap X$, a compact neighborhood of p in X . Let $K' = \tilde{\varphi}^{-1}(\tilde{L})$. The real part of K' is $K = \varphi^{-1}(L)$ which is compact since φ is proper. Let N be a compact neighborhood of K in \tilde{Y} which contains K and is preserved by the auto conjugation of \tilde{Y} . Let $\tilde{K} = K' \cap N$. The set \tilde{K} is a compact neighborhood in \tilde{Y} which is preserved by the auto conjugation of \tilde{Y} such that the real part of \tilde{K} is K . Let U be the real part of \tilde{U} which is an open neighborhood of p in X with closure L in X . Let $V = \varphi^{-1}(U)$, whose closure is $K = \varphi^{-1}(L)$. We have that

$$(16) \quad \varphi(V) = \varphi(K) \cap U.$$

For each $h \in \mathcal{E}_{\tilde{X}}$ such that $X_h \neq \emptyset$, we have associated complex analytic morphisms $\tilde{d}_h : \tilde{X}_h \rightarrow \tilde{X}$ with real part $d_h : X_h \rightarrow X$, and associated diagrams (12) with real part (14). For all i, j , we have that

$$d_h^{-1}(\varphi(K)) = d_h^{-1}(\varphi(K)) \cap [\cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(Y_{ij}))]$$

by (15). Thus

$$(17) \quad \begin{aligned} d_h^{-1}(\varphi(V)) &= d_h^{-1}(\varphi(K) \cap U) = d_h^{-1}(\varphi(K)) \cap d_h^{-1}(U) \\ &= d_h^{-1}(\varphi(K)) \cap [\cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(Y_{ij}))] \cap d_h^{-1}(U) \\ &= d_h^{-1}(U) \cap [\cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(Y_{ij}))]. \end{aligned}$$

We now establish that $d_h^{-1}(\varphi(V))$ is a semi analytic subset of X_h . For all i, j , $\varphi_{ij}(Y_{ij})$ is semi analytic in X_{ij} since φ_{ij} is a monomial morphism (by the Tarski Seidenberg theorem, c.f. Theorem 1.5 [8]). Thus $d_h^{-1}(\varphi(V))$ is semianalytic in X_h by (17).

For $h \in \mathcal{E}_{\tilde{X}}$, let \tilde{C}_h be an open relatively compact neighborhood of $h_{\tilde{X}}$ in \tilde{X}_h on which the auto conjugation acts. Let $\bar{d}_h : \tilde{C}_h \rightarrow \tilde{X}$ be the induced morphism. Let C be a compact neighborhood of p in \tilde{X} such that $C \subset \tilde{U}$ and let $C' = \rho_{\tilde{X}}^{-1}(C)$. The set C' is compact since $\rho_{\tilde{X}}$ is proper (Theorem 3.4 [21] or Theorem 3.16 [20]). The open sets $\mathcal{E}_{\tilde{d}_f}$ for $f \in C'$ give an open cover of C' , so there is a finite sub cover, which we index as $\mathcal{E}_{\tilde{d}_{f_1}}, \dots, \mathcal{E}_{\tilde{d}_{f_t}}$. We may replace \tilde{X}_{f_i} with $\tilde{d}_{f_i}^{-1}(\tilde{U})$, so that (17) implies that

$$d_h^{-1}(\varphi(Y)) = \cup_{i,j} \varepsilon_{ij}^{-1}(\varphi_{ij}(Y_{ij}))$$

is a semi analytic set.

Let C_{f_i} be the closure of \tilde{C}_{f_i} in $\tilde{X}_{\tilde{d}_{f_i}}$ which is compact. Since $\rho_{\tilde{X}}$ is surjective and continuous, we have inclusions of compact sets

$$p \in C \subset \cup_{i=1}^t \tilde{d}_{f_i}(C_{f_i}).$$

Since X and \tilde{X} are smooth and each \tilde{d}_{f_i} is a finite product of local blowups of closed analytic subspaces which are preserved by the auto conjugation, if F_{f_i} is the the union of the preimage in \tilde{X}_{f_i} of these subspaces, then F_{f_i} is a nowhere dense closed analytic subspace of \tilde{X}_{f_i} which is preserved by the auto conjugation such that $\tilde{X}_{f_i} \setminus F_{f_i} \rightarrow \tilde{X}$ is an open embedding. The image $\tilde{d}_{f_i}(F_{f_i})$ is nowhere dense in \tilde{X} , and since \tilde{X} and X are smooth varieties, $\tilde{d}_{f_i}(F_{f_i}) \cap X$ is nowhere dense in X (as in the proof of Theorem 8.7 [14]).

Let $C^* = C \cap X$ which is a compact neighborhood of p in X which is contained in L . Let $p' \in C^* \setminus \cup_{i=1}^t \tilde{d}_{f_i}(F_{f_i})$. Then there exist i and $e \in \mathcal{E}_{\tilde{d}_{f_i}}$ such that $e_{\tilde{X}} = p'$. Let $p_i = e_{\tilde{X}_{f_i}} \in C_i \subset \tilde{X}_{f_i}$. Since $p_i \notin F_{f_i}$, \tilde{d}_{f_i} is an open embedding near p_i , and since p' is real, $p_i \in X_{f_i}$ is real. Thus $p' \in d_{f_i}(C_{f_i} \cap X_{f_i})$. We thus have that the set $C^* \setminus \cup_{i=1}^t \tilde{d}_{f_i}(F_{f_i})$, which we have shown is dense in C^* , is contained in the compact set $\cup_{i=1}^t d_{f_i}(C_{f_i} \cap X_{f_i})$. Thus its closure C^* is contained in $\cup_{i=1}^t d_{f_i}(C_{f_i} \cap X_{f_i})$, giving the conclusion of 2) of the theorem. \square

Theorem 3.4. *Suppose that X is a smooth real analytic space. Suppose that A is a sub analytic subset of X and that $p \in X$. Then there exists a finite number of real analytic maps $\pi_\alpha : V_\alpha \rightarrow X$ such that:*

- 1) *Each π_α is a composition of local blowups of nonsingular sub varieties,*
- 2) *There exist compact neighborhoods N_α in V_α for all α such that $\cup_\alpha \pi_\alpha(N_\alpha)$ is a compact neighborhood of p in X ,*
- 3) *For all α , $\pi_\alpha^{-1}(A)$ is a semianalytic subset of V_α .*

Proof. After replacing X with a suitable open neighborhood of p , we have by Definition 6.10 [20], an expression

$$A = \cup_{k \in I} \cap_{l \in J} (A_{kl} \setminus B_{kl})$$

where I and J are nonempty finite index sets and there are proper real analytic maps of reduced analytic spaces $\varphi_{kl} : Y_{kl} \rightarrow X$ and $\psi_{kl} : Z_{kl} \rightarrow X$ such that $A_{kl} = \varphi_{kl}(Y_{kl})$ and $B_{kl} = \psi_{kl}(Z_{kl})$. Let \tilde{X} be a smooth complexification of X and let $\tilde{\varphi}_{kl} : \tilde{Y}_{kl} \rightarrow \tilde{X}$ and $\tilde{\psi}_{kl} : \tilde{Z}_{kl} \rightarrow \tilde{X}$ be complexifications of φ_{kl} and ψ_{kl} . For each $h \in \mathcal{E}_{\tilde{X}}$, $k \in I$ and $l \in J$ we construct as in the proof of Theorem 3.3 complex analytic morphisms of smooth analytic spaces $\tilde{d}_h^{kl} : (\tilde{X}_h)_{kl} \rightarrow \tilde{X} \in h$ and $(\tilde{d}'_h)^{kl} : (\tilde{X}'_h)_{kl} \rightarrow \tilde{X} \in h$ with auto conjugations, so that taking the invariants of the auto conjugations we have real analytic morphisms $d_h^{kl} : (X_h)_{k,l} \rightarrow X$ and $(d'_h)^{kl} : (X'_h)_{kl} \rightarrow X$ such that $(d_h^{kl})^{-1}(A_{kl})$ is semianalytic in $(X_h)_{kl}$

and $((d'_h)^{kl})^{-1}(B_{kl})$ is semianalytic in $(X'_h)_{kl}$ for $k \in I$ and $l \in J$. There exists $\tilde{d}_h : \tilde{X}_h \rightarrow \tilde{X} \in h$ with auto conjugation such that there are factorizations $\tilde{\beta}_{kl} : \tilde{X}_h \rightarrow (\tilde{X}_h)_{kl}$ and $\tilde{\gamma}_{kl} : \tilde{X}_h \rightarrow (\tilde{X}'_h)_{kl}$ for all $k \in I$ and $l \in J$. Thus taking the real part of $\tilde{d}_h : \tilde{X}_h \rightarrow \tilde{X}$, we have real analytic morphisms $\beta_{kl} : X_h \rightarrow (X_h)_{kl}$ and $\gamma_{kl} : X_h \rightarrow (X'_h)_{kl}$ factoring through $X_h \rightarrow X$. Thus

$$d_h^{-1}(A_{kl}) = \beta_{kl}^{-1}(d_h^{kl})^{-1}(A_{kl})$$

and

$$d_h^{-1}(B_{kl}) = \gamma_{kl}^{-1}((d'_h)^{kl})^{-1}(B_{kl})$$

are semi analytic in X_h .

We now proceed as in the proof of Theorem 3.3 to obtain the condition 2) of Theorem 3.4. \square

Theorem 3.5. *Let X be a smooth connected real analytic space and let A be a sub analytic subset of X . Let $p \in X$ and let $n = \dim X$. Then there exist a finite number of real analytic morphisms $\pi_\alpha : V_\alpha \rightarrow X$ which are finite sequences of local blowups over X and induce an open embedding of an open dense subset of V_α into X such that:*

- 1) *Each V_α is isomorphic to \mathbb{R}^n ,*
- 2) *There exist compact neighborhoods K_α in V_α such that $\cup_\alpha(K_\alpha)$ is a compact neighborhood of p in X ,*
- 3) *For each α , $\pi_\alpha^{-1}(A)$ is union of quadrants in \mathbb{R}^n .*

Proof. The proof follows from Theorem 3.4 and Proposition 7.2 [20] and Lemma 7.2.1 [20]. \square

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